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# On a Theorem of Titchmarch–Kodaira–Weidmann Concerning Absolutely Continuous Operators, I

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## 1. INTRODUCTION

In his book [3] Titchmarsh formulated a condition on the potential $p$ which ensured that the part of the operator $-D^2 + p$ over the open positive axis is absolutely continuous. Specifically he showed that this holds for an appropriate self-adjoint extension of this operator provided that $p$ is absolutely integrable and satisfies some very general conditions. For brevity we shall refer to such potentials as Titchmarsh potentials. At the same time Kodaira [4] studied this problem independently. He showed that this property holds for a different class of potentials, namely, those for which the derivative satisfies the estimate,

$$p'(\xi) = O\Big(\frac{1}{1+\xi}\Big)^{1+\epsilon}, \qquad \epsilon > 0, \text{ for } \xi \to \infty.$$

Later Weidmann [9] took up this problem and gave a common generalization of both of these results. His result implies that this holds for potentials which can be written as a sum of a Titchmarsh potential and another one which tends to zero at infinity and has an absolutely integrable derivative. Weidmann's result is quite remarkable, for potentials of the form,

$$p(\xi) = \sin \xi/\xi + \alpha \cos \xi/\xi$$

barely fail to satisfy these conditions. At the same time, according to Wigner and von Neumann [1] and Simon [11], for such potentials the operator $-D^2 + p$ does admit positive point eigenvalues.

In this paper we give another proof of this special Weidmann result.

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Specifically we illustrate how the JWKB approximation method can be applied to verify a set of abstract criteria for absolute continuity, to be formulated presently.

These criteria are based on a simple abstract property which has been employed by several different authors to establish absolute continuity. In particular it has been employed by Kato and Kuroda [14], Agmon [16], Lavine [21], Combes [23], Kuroda [17], Hoegh-Krohn [37], and elsewhere [8]. It is an abstract version of the limiting absorption principle [7, 13, 18].
It is an abstract version of the limiting absorption principle [7, 13, 18].
It is implied, under general circumstances, by the existence of an eigenfunction expansion [3, 10, 12]. The set of criteria itself is an abstract version of the Lippman–Schwinger equations [25, 31]. For a different abstract version of the Lippman–Schwinger equations we refer to the work of Amrein, Georgescu, and Jauch [15] and Prugovecki [22].

In Section 2 we first give a precise description of this special Titchmarsh–Kodaira–Weidmann class of potentials. This is done with the aid of Condition S(T–K–W). Then in Theorem 2.1 we formulate the previously mentioned absolute continuity statement about the operator $-D^2 + p$.

In Section 3 we formulate a set of criteria for the absolute continuity of the part of a given self-adjoint operator $A$ over a given interval.

First in Lemma 3.1 we give a set of preliminary criteria. Then in Theorem 3.1 we formulate these criteria. Finally in Lemma 3.3 we isolate a property which is basic in verifying the assumptions of Theorem 3.1. Again this lemma is a version of a result used by several authors.

In Section 4 we derive Theorem 2.1 from Theorem 3.1. First in Lemmas 4.1 and 4.2 we construct a factorization of the resolvent of this operator and a norm such that this factorization satisfies the assumptions of Theorem 3.1 with reference to this norm. This construction is based on the JWKB approximation method [25, 31]. So to speak, this approximation allows us to take the effect of the tail of the long range potential into consideration. Second, in Lemmas 4.3 and 4.4 we establish the assumptions of Theorem 3.1 for this factorization. The proofs of these two lemmas are similar to the proofs employed for short range potential perturbations. In fact we exploit an elegant observation of Kato [29b] saying that for the case of 1-space dimension the Hilbert–Schmidt norm can be employed to study smooth or gentle perturbations.

It is a pleasure to thank Professor Sibuya for introducing the author to the theory of the JWKB approximation. Special thanks are due to Professor Lavine for an informal conversation at the recent Scattering Theory Summer School in Denver. During this conversation he emphasized that the construction of an energy dependent perturbation is one of the essential features of his theory [24]. This suggested the emphasis of a similar construction in the report [19]. This, in turn, led to the present Section 3.

After this work was completed we learned about the overlapping and deep results of Georgescu [34].

## 2. Formulation of the Result

We start this section by stating a condition which is somewhat more restrictive than the one of Titchmarsh [3], Kodaira [4], and Weidmann [9] mentioned in the introduction. In the formulation of this special Titchmarsh–Kodaira–Weidmann condition $\mathscr{R}^+$ denotes the open axis $(0, \infty)$.

Condition S(T–K–W). The potential $p$ is a real valued function on $\mathscr{R}^+$ and it can be written in the form

$$p = p_1 + p_2, \tag{2.1}$$

where both $p_1$ and $p_2$ are real. The potential $p_2$ is such that

$$\lim_{\xi\to\infty} p_2(\xi) = 0 \tag{2.2}$$

and

$$\int_0^\infty p_2'(\xi)\, d\xi < \infty. \tag{2.3}$$

The potential $p_1$ is such that

$$p_1 \in \mathfrak{L}_1(\mathscr{R}^+). \tag{2.4}$$

Next assume that the potential $p$ satisfies this condition and with the aid of it define $\mathfrak{D}(L(p))$ to be the set of those functions $f$ in $\mathfrak{L}_2(\mathscr{R}^+)$ which have absolutely continuous first derivatives and for which

$$f(0) = 0 \qquad \text{and} \qquad -f'' + pf \in \mathfrak{L}_2(\mathscr{R}^+).$$

Then define the operator $L(p)$ mapping $\mathfrak{D}(L(p))$ into $\mathfrak{L}_2(\mathscr{R}^+)$ by the equation

$$L(p)f = -f'' + pf.$$

For convenience we also assume that $p$ is locally square integrable and

$$\lim_{\xi\to\infty} \int_\xi^{\xi+1} p^2(\eta)\, d\eta = 0. \tag{2.7}$$

Then denoting differentiation by $D$, as is well known [29, 32] this implies that $M(p)$, the operator of multiplication by the potential $p$, is $-D^2$ compact. This, in turn, implies [29, 32] that

$$\mathfrak{D}(L(p)) = \mathfrak{D}(L(0)) = \mathfrak{D}(-D^2),$$

and hence

$$L(p) = -D^2 + M(p). \tag{2.9}$$

THEOREM 2.1. *Suppose that the potential p satisfies Condition S(T-K-W) and define the operator L(p) by Eqs.* (2.5) *and* (2.9). *Then the part of the operator L(p) over* $\mathscr{R}^+$ *is absolutely continuous, that is,*

$$L(p)(\mathscr{R}^+) = L(p)(\mathscr{R}^+)_{ac}. \tag{2.10}$$

In Section 4 we shall derive this theorem from an abstract theorem to be stated in Section 3.

## 3. AN ABSTRACT CRITERION FOR ABSOLUTE CONTINUITY

Let $A$ be a given self-adjoint operator acting on a given abstract Hilbert space $\mathfrak{H}$. In this section we formulate two sets of conditions for a part of $A$ to be absolutely continuous.

To a given interval of reals, $\mathscr{I}$, and angle $\alpha$, we assign two open regions of the complex plane by setting

$$\mathscr{R}_{\pm}(\mathscr{I}) = \{\mu : \operatorname{Re} \mu \in \mathscr{I}^0, 0 < \pm \arg \mu < \alpha\}, \tag{3.1}$$

where $\mathscr{I}^0$ denotes the interior of the interval $\mathscr{I}$. As usual, we denote by $\mathfrak{B}(\mathfrak{H})$ the space of everywhere defined bounded operators on $\mathfrak{H}$. For a possibly unbounded operator $T$ and for $\mu$ in $\rho(T)$, the resolvent set of $T$, we set

$$R(\mu; T) = (\mu I - T)^{-1} \in (\mathfrak{B}).$$

Next suppose that $\mathfrak{G}$ is a Banach space such that both $\mathfrak{G}$ and $\mathfrak{H}$ can be embedded in a metric space $\mathfrak{M}$ in such a manner that $\mathfrak{G} \cap \mathfrak{H}$ is dense in each of the spaces $\mathfrak{G}$ and $\mathfrak{H}$. We shall refer to this property as the dense intersection property. In view of this fact each operator $A$ on $\mathfrak{G} \cap \mathfrak{H}$ defines a form on $\mathfrak{G} \cap \mathfrak{H} \times \mathfrak{G} \cap \mathfrak{H}$, namely, the form defined by

$$[A]_{\mathfrak{G}}(f, g) = [A]_{\mathfrak{H}}(f, g) = (Af, g).$$

If this form is bounded with reference to the $\mathfrak{G}$-norm, we denote its closure by the same symbol $[A]_{\mathfrak{G}}$ and say that the operator $A$ determines a bounded form on $\mathfrak{G}$. Note that the boundedness of this form does not imply and is not implied by the property that $A$ is in $\mathfrak{B}(\mathfrak{G})$. Similarly, if the operator $A$ maps $\mathfrak{G} \cap \mathfrak{H}$ into itself and it is bounded with reference to the $\mathfrak{G}$-norm, we denote its closure by $A_0$, and say that $A$ determines an operator in $\mathfrak{B}(\mathfrak{G})$.

With the aid of these notations we formulate the following condition.

CONDITION $G(\mathscr{I})$. For each $\mu$ in $\mathscr{R}_+(\mathscr{I})$ the given factorization of the resolvent

$$R(\mu) = S_0(\mu) \cdot Q(\mu), \qquad S_0(\mu), Q(\mu) \in \mathfrak{B}(\mathfrak{H}) \tag{3.3}$$

is such that the first factor satisfies Condition $G_1(\mathscr{I})$ and the second factor satisfies Condition I.$\mathscr{I}$ that follow.

CONDITION $G_1(\mathscr{I})$. For each $\mu$ in $\mathscr{R}_\pm(\mathscr{I})$ the operator $S_0(\mu)$ on $\mathfrak{H}$ determines a bounded form on $\mathfrak{G} \times \mathfrak{G}$ for which

$$\sup_{\mu \in \mathscr{R}_\pm(\mathscr{I})} \| [S_0(\mu)]_{\mathfrak{G}} \| < \infty. \tag{3.4}$$

CONDITION I.$\mathscr{I}$. For each $\mu$ in $\mathscr{R}_\pm(\mathscr{I})$ the operator $Q(\mu)$ on $\mathfrak{H}$ determines an operator in $\mathfrak{B}(\mathfrak{G})$ for which

$$\sup_{\mu \in \mathscr{R}_\pm(\mathscr{I})} \| Q(\mu)_{\mathfrak{G}} \| < \infty. \tag{3.5}$$

The following lemma employs Condition $G(\mathscr{I})$ and it is our first absolute continuity criterion.

LEMMA 3.1. *Let $\mathscr{I}$ be a compact interval. Suppose that $R(\mu; A)$ admits a factorization which satisfies Condition $G(\mathscr{I})$ with reference to a space $\mathfrak{G}$ having the dense intersection property. Then $A(\mathscr{I})$, the part of $A$ over the interval $\mathscr{I}$, is absolutely continuous.*

The resolvents of a large class of Schrodinger operators admit a factorization with reference to a given norm $\mathfrak{G}$ such that the inverses of the operators $Q(\mu)$ satisfy the following two conditions.

CONDITION $G_2(\mathscr{I})$. For each $\mu$ in the open regions $\mathscr{R}_\pm(\mathscr{I})$ the operator $Q(\mu)$ in $\mathfrak{H}$ admits an inverse in $\mathfrak{B}(\mathfrak{H})$. This inverse determines an operator in $\mathfrak{B}(\mathfrak{G})$ and this family of inverses, $Q(\mu)_{\mathfrak{G}}^{-1}$, depends norm-continuously on $\mu$. Furthermore it admits continuous extensions onto the closures $\overline{\mathscr{R}_\pm(\mathscr{I})}$.

CONDITION $A(\mathscr{I})$. For each $\omega$ in the compact interval $\mathscr{I}$ each of the two limit operators, $Q^+(\omega)_{\mathfrak{G}}^{-1}$, admit inverses in $\mathfrak{B}(\mathfrak{G})$.

It is not difficult to show that Condition $G_2(\mathscr{I})$ and Condition $A(\mathscr{I})$ together imply Condition I.$\mathscr{I}$. Inserting this implication in Lemma 3.1 yields the following theorem, which is our second absolute continuity criterion.

THEOREM 3.1. *Let $A$ be a given self-adjoint operator and let $\mathscr{I}$ be a compact interval. Suppose that $R(\mu\ A)$ admits a factorization which satisfies Conditions*

*$G_{1,2}(\mathscr{I})$ and Condition $A(\mathscr{I})$ with reference to a space $\mathfrak{G}$ having the dense intersection property. Then $A(\mathscr{I})$, the part of the operator $A$ over the interval $\mathscr{I}$, is absolutely continuous.*

For a large class of Schrödinger operators for each $\omega$ in $\mathscr{I}$ each of the two limit operators, $Q^{\pm}(\omega)_{\mathfrak{G}}^{-1}$, is Fredholm of index zero [36]. That is to say invertibility is implied by the one to one property. We shall call the point $\omega$ exceptional if either of the two limit operators, $Q^{\pm}(\omega)_{\mathfrak{G}}^{-1}$, is not one to one. The following condition will allow us to formulate a useful property of exceptional points.

CONDITION $G_3(\mathscr{I})$. ($\alpha$) To each $\omega$ in $\mathscr{I}$ there is a self-adjoint operator $A_0(\omega)$ such that $\mathfrak{D}(A_0(\omega)) \supset \mathfrak{D}(A)$ and the resolvent

$$R_0(\mu) = R(\mu; A_0(\operatorname{Re}\mu)) \tag{3.6}$$

satisfies Condition $G_1(\mathscr{I})$.

($\beta$) The limits of the forms $[R_0(\mu)]_{\mathfrak{G}}$ and $[S_0(\mu)]_{\mathfrak{G}}$ exist and are equal. Specifically

$$\lim_{\mu\to\omega} [S_0(\mu)]_{\mathfrak{G}} = \lim_{\mu\to\omega} [R_0(\mu)]_{\mathfrak{G}} , \tag{3.7}$$

uniformly in $\omega$ in $\mathscr{I}$.

($\gamma$) For each $\mu$ in $\mathscr{R}_{\pm}(\mathscr{I})$ the operator

$$T(\mu) = (A - A_0(\operatorname{Re}\mu))R_0(\mu) \in \mathfrak{B}(\mathfrak{H})$$

determines an operator in $\mathfrak{B}(\mathfrak{G})$ such that

$$\lim_{\mu\to\omega} \| (I - T(\mu))_{\mathfrak{G}} - Q(\mu)_{\mathfrak{G}}^{-1} \| = 0, \tag{3.9}$$

uniformly in $\omega$ in $\mathscr{I}$.

Incidentally note that for a perturbation $(A_0 , A - A_0)$, which is gentle over the interval $\mathscr{I}$ [8, 37], Conditions $G_{1,2}(\mathscr{I})$ hold with

$$S_0(\mu) = R(\mu; A_0), \qquad Q(\mu) = [I - (A - A_0)R(\mu; A_0)]^{-1}.$$

Furthermore setting

$$A_0(\omega) = A_0 , \qquad R_0(\mu) = R(\mu; A_0), \qquad T(\mu) = (A - A_0)R(\mu; A_0),$$

we see that Conditions $G_{1,2}(\mathscr{I})$ imply Condition $G_3(\mathscr{I})$.

The basic lemma that follows describes a useful property of exceptional points. It is an abstract version of a result of Povzner [5] and Ikebe [6]. Various versions of this result have been formulated by various authors

[10, 14, 16] for various classes of operators. The present version, which was proved in the technical report [38], is similar to the one of Lavine [21] and to the one formulated elsewhere [8b]. In it we set

$$T^{\pm}(\omega)_{\mathfrak{G}} = \lim_{\epsilon \to +0} T(\omega \pm i\epsilon)_{\mathfrak{G}}\,, \tag{3.10$^\pm$}$$

whenever this limit exists in $\mathfrak{B}(\mathfrak{G})$.

LEMMA 3.3. *Suppose that the point $\omega$ in $\mathscr{I}$ and the vector $h$ in $\mathfrak{G}$ are such that for the operators of definitions* (3.9) *and* (3.10)$^{\pm}$

$$(I - T^{\pm}(\omega))_{\mathfrak{G}} h = 0 \qquad of \qquad (I - T^{\pm}(\omega))_{\mathfrak{G}} h = 0. \tag{3.11$^\pm$}$$

*Then*

$$[S_0^{+}(\omega)_{\mathfrak{G}}(h, h) - [S_0^{-}(\omega)]_{\mathfrak{G}}(h, h) = 0. \tag{3.12}$$

## 4. THE PROOF OF THEOREM 2.1

In this section we derive Theorem 2.1 from the abstract Theorem 3.1 by constructing a factorization of the resolvent satisfying the conditions of this abstract theorem. We start this construction by imposing an additional condition on the potential which will allow the application of the JWKB approximation method. Roughly speaking the condition that follows will hold for the smoothed out tail of the long range part of $p$.

CONDITION $O(\mathscr{I})$. The potential $p$, is real valued and twice continuously differentiable. Furthermore

$$\sup_{\mu \in \mathscr{R}_{\pm}(\mathscr{I})} \sup_{\xi \in \mathscr{R}^{+}} \left| \frac{1}{\mu - p(\xi)} \right| < \infty \tag{4.1}$$

and $Dp$ and $D^2p$ are bounded functions such that

$$\int_0^{\infty} (| Dp(\xi)|^2 + | D^2p(\xi)|)\, d\xi < \infty. \tag{4.2}$$

LEMMA 4.1. *Let $\mathscr{I}$ be a compact subinterval of $\mathscr{R}^+$ and suppose that the potential $p$ satisfies Condition $S(T$–$K$–$W)$. Then it admits a decomposition of the form,*

$$p = p_1 + p_2\,, \tag{4.3}$$

*where $p_1$ is short range in the sense that*

$$p_1 \in \mathfrak{L}_1(\mathscr{R}^{+}) \tag{4.4}$$

*and $p_2$ satisfies Condition $O(\mathscr{I})$.*

To establish this lemma, clearly, it is no loss of generality to assume that $p$ is long range. To construct such a $p_2$ we first mollify $p$. More specifically, let $j$ be a twice continuously differentiable positive function with support in $(0, 1)$ such that

$$\int_0^\infty j(\eta)\, d\eta = 1. \tag{4.5}$$

We maintain that setting

$$Jp(\xi) = \int_0^\infty j(\eta - \xi)\, p(\eta)\, d\eta; \tag{4.6}$$

we have

$$p - Jp \in \mathfrak{L}_1(\mathscr{R}^+) \tag{4.7}$$

and

$$(DJp)^2 \in \mathfrak{L}_1(\mathscr{R}^+) \qquad \text{and} \qquad D^2Jp \in \mathfrak{L}_1(\mathscr{R}^+). \tag{4.8}$$

To establish relation (4.7) note that relations (4.5) and (4.6) together with the fundamental theorem of the calculus show that

$$p(\xi) - Jp(\xi) = -\int_0^\infty j(\eta - \xi) \int_\eta^\xi Dp(\zeta)\, d\zeta\, d\eta. \tag{4.9}$$

This yields

$$|p(\xi) - Jp(\xi)| \leq \int_\xi^{\xi+1} |Dp(\zeta)|\, d\zeta, \tag{4.10}$$

if we remember that the support of the function $j$ is in the interval $(0, 1)$ and combine relation (4.5) with the translation invariance of the Lebesgue measure. It is an elementary fact that

$$\int_0^\infty \int_\xi^{\xi+1} |Dp(\zeta)|\, d\zeta\, d\xi = \int_0^1 \int_\eta^\infty |Dp(\zeta)|\, d\zeta\, d\eta. \tag{4.11}$$

According to Condition S(T–K–W) the right member is finite. Inserting this fact and relation (4.11) in estimate (4.10) we obtain the validity of relation (4.7). To establish relation (4.8) note that the support of the derivatives of $j$ are also in the interval $(0, 1)$. Hence differentiating (4.5) under the integral and integrating by parts yields

$$DJp(\xi) = \int_0^\infty j(\eta - \xi)\, Dp(\eta)\, d\eta$$

and

$$D^2Jp(\xi) = \int_0^\infty Dj(\eta - \xi) \cdot Dp(\eta)\, d\eta.$$

Inserting assumption (2.3) in these formulas we see that each of these two functions is bounded. Integrating these formulas with reference to the variable $\xi$ we obtain that each of these functions is in $\mathfrak{L}_1(\mathscr{R}^+)$. This establishes the validity of relation (4.7) if we remember that $DJp$ is also bounded.

Second let $a$ be a twice continuously differentiable function such that

$$a(\xi) = \begin{cases} 0 & 0 \leqslant \xi \leqslant 1 \\ 1 & \xi \geqslant 2, \end{cases}$$

and define the sequence of functions $a^n$ by

$$a^n(\xi) = a(\xi/n).$$

Then clearly the function $a^n$ vanishes in the interval $[0, n]$ and for the $\mathfrak{L}_1(\mathscr{R}^+)$-norms, we have

$$\lim_{n\to\infty} \| a^n J_p \|_1 = \lim_{n\to\infty} \| a^n DJp \|_1 = \lim_{n\to\infty} \| a^n D^2 Jp \|_1 = 0.$$

Hence for sufficiently large $n$, the function

$$p_2 = a^n Jp \tag{4.12}$$

satisfies Condition $O(\mathscr{I})$. At the same time the validity of conclusion (4.3) follows from relation (4.7). This completes the proof of Lemma 4.1.

LEMMA 4.2. *Suppose that the potential $p_2$ satisfies Condition $O(\mathscr{I})$ and for each nonreal $\mu$ define the potential $q(\mu)$ by*

$$q(\mu)(\xi) = p_2 + \frac{1}{4}\frac{D^2 p_2(\xi)}{\mu - p_2(\xi)} + \frac{5}{16}\left(\frac{Dp_2(\xi)}{\mu - p_2(\xi)}\right)^2. \tag{4.13}$$

*Then*

$$\mu \in \rho(L(q(\mu))). \tag{4.14}$$

To illustrate the proof of conclusion (4.13) we first observe that the JWKB approximation method allows us to give this resolvent kernel explicitly. Specifically, set

$$y^{\pm}(\mu)(\xi) = \left(\frac{1}{\mu - p_0(\xi)}\right)^{1/4} \exp\left[\pm i \int_0^{\xi} (\mu - p_0(\sigma))^{1/2}\, d\sigma\right]. \quad (4.15)^{\pm}$$

Here we denote by $+\mu^{1/2}$ that particular branch of this doubly valued function for which

$$\mathrm{Re}(+i(\mu)^{1/2}) < 0 \qquad \text{and hence} \qquad y^+(\mu) \in \mathfrak{L}_2(\mathscr{R}), \qquad (4.16)^+$$

for each nonreal $\mu$. Next define two more functions, by setting

$$y_0(\mu) = y^+(\mu) - y^-(\mu) \qquad (4.17)_0$$

and

$$y_\infty(\mu) = y^+(\mu). \qquad (4.17)_\infty$$

Then, as is well known [25, 31], these definitions imply that each of these two functions satisfies the differential equation

$$D^2 y(\mu)(\xi) + [\mu - q(\mu)(\xi)] y(\mu)(\xi) = 0. \tag{4.18}$$

In fact, this property motivated our definition (4.13). According to definitions $(4.17)_{0,\infty}$ the function $y_0(\mu)$ satisfies the boundary condition (2.5) at $\xi = 0$ and the function $y_\infty(\mu)$ is square integrable at $\xi = \infty$. As is well known [30] conditions more general than our Condition $O(\mathscr{I})$ imply that these properties determine the functions $y_{0,\infty}(\mu)$ up to a constant multiple. These facts allow us to apply, at least formally, the Weyl representation theorem for the resolvent kernel of a self-adjoint differential operator [30]. This yields

$$R(\mu; L(q(\mu)))(\xi, \eta) = \frac{1}{W(y_0(\mu), y_\infty(\mu))} \begin{cases} y_0(\mu)(\xi)\, y_\infty(\mu)(\eta), & \text{for } \eta \geq \xi \\ y_\infty(\mu)(\xi)\, y_0(\mu)(\eta), & \text{for } \eta \leq \xi, \end{cases} \tag{4.19}$$

where the denominator of the first factor is the Wronskian of these solutions. Note that this application of the Weyl representation theorem is formal inasmuch as the operator $L(q(\mu))$ is not self-adjoint, in fact, not even symmetric. Nevertheless it is not difficult to show that this kernel defines an operator in $\mathfrak{B}(\mathfrak{L}_2(\mathscr{R}^+))$ which is the inverse of the operator $\mu I - L(q(\mu))$. For the details of this proof we refer to the technical report [38] and consider the proof of Lemma 3.2 complete.

In the next lemma we use the approximate resolvent $R(\mu; L(q(\mu)))$ to define a factorization of the original resolvent.

Lemma 4.3. *Suppose that the potential $p$ satisfies Condition $S(T\text{-}K\text{-}W)$ and for each nonreal $\mu$ define the potential $q(\mu)$ by Lemma 4.2. Then*

$$[I - M(p - q(\mu)) R(\mu; L(q(\mu)))]^{-1} \in \mathfrak{B}(\mathfrak{H}). \tag{4.20}$$

*This operator,*

$$Q(\mu) = [I - M(p - q(\mu)) R(\mu; L(q(\mu)))]^{-1}, \tag{4.21}$$

*defines a factorization of the resolvent of the operator $L(p)$ of definition (2.9). Specifically*

$$R(\mu; L(p)) = R(\mu; L(q(\mu))) \cdot Q(\mu). \tag{4.22}$$

The proof of this lemma is not difficult either and for the details we refer to the technical report [38].

In the remainder of this section we set

$$S(\mu) = R(\mu; L(q(\mu))). \tag{4.23}$$

Then we show that with this choice the factorization (4.22) of Lemma 4.3 satisfies the rather stringent conditions of the abstract Theorem 3.1. To construct such a space $\mathfrak{G}$ first with the aid of the potentials $p$ of Theorem 2.1 and $q(\mu)$ of Lemma 4.2 we define a function by setting

$$n(\xi) = \sup_{\mu \in \mathscr{R}_{\pm}(\mathscr{I})} |p(\xi) - q(\mu)(\xi)| + \exp(-\xi). \tag{4.24}$$

Then define the space $\mathfrak{G}$ to consist of those functions $f$ in $\mathfrak{H} = \mathfrak{L}_2(\mathscr{R}^+)$ for which the norm

$$\|f\|_{\mathfrak{G}} = \| M(1/n)^{1/2} f \|_{\mathfrak{H}} \tag{4.25}$$

is finite. This norm is similar to the smoothness norm of Kato and Kuroda [14, 17] inasmuch as it defines a Hilbert space and, in turn, it is defined with the aid of a factorization.

(a) Condition $G_1(\mathscr{I})$. To verify that for the operator of definition (4.23) this condition holds we first need a fact observed elsewhere. Specifically, we need that

$$\| [S_0(\mu)]_{\mathfrak{G}} \| \leqslant \| M(n^{1/2}) S_0^{\bullet}(\mu) M(n^{1/2}) \|_{\mathfrak{H}} . \tag{4.26}$$

Hence the validity of Condition $G_1(\mathscr{I})$ is implied by

$$\sup_{\mu \in \mathscr{R}_{\pm}(\mathscr{I})} \| M(n^{1/2})\, S_0(\mu)\, M(n^{1/2}) \|_{\mathfrak{H}} < \infty. \tag{4.27}$$

We start the proof of this estimate by formulating a technical lemma.

LEMMA 4.4. *Suppose that the potential $p_2$ satisfies Condition $O(\mathscr{I})$ and define the function $q(\mu)$ by Lemma* 4.2. *Then the kernel of the operator $S_0(\mu)$ of definition* (4.23) *is bounded. Specifically*

$$\sup_{\mu \in \mathscr{R}_{\pm}(\mathscr{I})} \ \sup_{(\xi,\eta) \in \mathscr{R}^+ \times \mathscr{R}^+} | S_0(\mu)(\xi, \eta)| < \infty. \tag{4.28}$$

*Furthermore as $\mu$ approaches the point $\omega$ of $\mathscr{R}^+$ the kernel $S_0(\mu)(\xi, \eta)$ converges, uniformly in $(\xi, \eta)$ on any compact subset of $\mathscr{R}^+ \times \mathscr{R}^+$.*

This lemma is an elementary consequence of formula (4.19) and for a detailed proof of this fact we refer to the technical report [38].

Next we derive estimate (4.27) from Lemma 4.4. For this purpose recall Lemma 4.1. It shows that the function $n$ of definition (4.24) is integrable, that is,

$$n \in \mathfrak{L}_1(\mathscr{R}^+).$$

This, in turn, together with Lemma 4.4 shows that the kernel of the operator in estimate (4.27) is square integrable and hence is Hilbert–Schmidt. At the same time it follows that this Hilbert–Schmidt norm is bounded uniformly in $\mu$ in $\mathscr{R}_{\pm}(\mathscr{I})$. This establishes the validity of estimate (4.27). This, in turn establishes the validity of Condition $G_1(\mathscr{I})$.

(b) Condition $G_2(\mathscr{I})$. Definitions (4.21) and (4.23) together show that

$$Q(\mu)^{-1} - I = M(p - q(\mu)) \in \mathfrak{B}(\mathfrak{H}). \tag{4.30}$$

To see that this operator determines an operator in $\mathfrak{B}(\mathfrak{G})$ recall definition (4.25). It shows that $M(1/n)^{1/2}$ is an isometry mapping $\mathfrak{G}$ onto all of $\mathfrak{H}$. That is to say it is a unitary transformation and we have the following unitary equivalence,

$$M(p - q(\mu))\, S_0(\mu)_{\mathfrak{G}} \sim M(1/n)^{1/2}\, M(p - q(\mu))\, S_0(\mu)\, M(n)^{1/2}_{\mathfrak{H}}. \tag{4.31}$$

Since unitarily equivalent operators have the same norm [30], this yields

$$\| M(p - q(\mu))\, S_0(\mu)_{\mathfrak{G}} \| = \| M(1/n)^{1/2}\, M(p - q(\mu))\, S_0(\mu)\, M(n)^{1/2}_{\mathfrak{H}} \|. \tag{4.32}$$

Note that this relation is analogous but different from relation (4.26) inasmuch as the first factor on the right is $M(1/n)^{1/2}$. For brevity set

$$X(\mu) = M(1/n)^{1/2}\, M(p - q(\mu)) S_0(\mu) M(n)^{1/2}. \tag{4.33}$$

Then formula (4.19) and definition (4.23) show that $X(\mu)$ is an integral operator. At the same time it follows from Lemma 4.4 and definition (4.24) and relation (4.29) that its Hilbert–Schmidt norm is bounded, in fact, uniformly so in $\mu$ in $\mathscr{R}_{\pm}(\mathscr{I})$. Hence

$$\sup_{\mu \in \mathscr{R}_{\pm}(\mathscr{I})} \| X(\mu) \|_{\mathfrak{H}} < \infty. \tag{4.34}$$

Similarly it follows that

$$\lim_{\mu_1 \to \omega} \lim_{\mu_2 \to \omega} \| X(\mu_1) - X(\mu_2) \|_{\mathfrak{H}} = 0.$$

In fact, this limit holds with reference to the Hilbert–Schmidt norm and it is uniform in $\omega$ in $\mathscr{I}$. Combining this relation with the equivalence relation (4.31) and definition (4.33) we obtain the validity of the second half of Condition $G_2(\mathscr{I})$.

(c) Condition $G_3(\mathscr{I})$. To verify Condition $G_3(\mathscr{I})$ ($\alpha$) recall definition (4.13) and that by assumption the potential $p_2$ satisfies Condition $O(\mathscr{I})$. They show that as $\mu$ approaches a given point $\omega$ of $\mathscr{I}$ the function $q(\mu)$ approaches a limit. At the same time it follows that this limit is the same as $\mu$

approaches $\mathscr{C}$ through the points of $\mathscr{R}_+(\omega)$ or $\mathscr{R}_-(\omega)$. This fact allows us to define the self-adjoint operator $A_0(\omega)$ by

$$A_0(\omega) = L(q_0(\omega)). \tag{4.35}$$

Inserting definition (4.35) in definition (3.6) yields

$$R_0(\mu) = R(\mu, L(q(\operatorname{Re} \mu))). \tag{4.36}$$

According to Section 4a Condition $G_1(\mathscr{I})$ holds for the family of operators $S_0(\mu)$. Hence Condition $G_3(\mathscr{I})(\alpha)$, that is to say Condition $G_1(\mathscr{I})$ for the family $R_0(\mu)$, is implied by

$$\lim_{\mu\to\omega} \| [R_0(\mu)]_{\mathfrak{S}} - [S_0(\mu)]_{\mathfrak{S}} \| = 0, \tag{4.37}$$

uniformly in $\omega$ in $\mathscr{I}$.

To illustrate the proof of this key relation first we show that, at least formally,

$$R_0(\mu) - S_0(\mu) = S_0(\mu)[I - M(q(\operatorname{Re} \mu) - q(\mu))S_0(\mu)]^{-1} \times M(q(\operatorname{Re} \mu) - q(\mu))S_0(\mu). \tag{4.38}$$

Application of the second resolvent equation to the operators $L(q(\mu))$ and $L(q(\operatorname{Re}(\mu)))$ at the point $\mu$ yields

$$R(\mu, L(q(\mu))) - R(\mu, L(q(\operatorname{Re} \mu))) = R(\mu, L(q(\mu)))(L(q(\mu)) - L(q(\operatorname{Re} \mu)))\, R(\mu, L(q(\operatorname{Re} \mu))).$$

According to relation (2.9)

$$L(q(\mu)) - L(q(\operatorname{Re} \mu) = M(q(\mu) - q(\operatorname{Re} \mu)).$$

Combining these two relations with relation (4.36) and definition (4.23) we obtain

$$R_0(\mu) - S_0(\mu) = R_0(\mu)(M(q(\mu) - q(\operatorname{Re} \mu))S_0(\mu). \tag{4.39}$$

Expressing $R_0(\mu)$ in terms of $S_0(\mu)$ from this equation and inserting the result into the same equation we arrive at relation (4.38). According to the technical report [38] Lemma 4.1 and a version of the Rellich–Kato theorem allows us to make the present formal argument rigorous. This completes the proof of relation (4.38).

Next set

$$Y(\mu) = M(1/n)^{1/2}\, M(q(\operatorname{Re} \mu) - q(\mu))S_0(\mu)M(n)^{1/2} \tag{4.40}$$

and

$$W(\mu) = M(1/n)^{1/2}[I - M(q(\operatorname{Re} \mu) - q(\mu))S_0(\mu)]^{-1}M(n)^{1/2}. \tag{4.41}$$

More specifically we define these operators to be the closures of the triple products on the left. Inserting these definitions in formula (4.38) we obtain

$$M(n)^{1/2}(R_0(\mu) - S_0(\mu))M(n)^{1/2} = M(n)^{1/2}S_0(\mu)M(n)^{1/2}W(\mu)Y(\mu). \qquad (4.42)$$

To illustrate the proof of relation (4.37), secondly we note that

$$\lim_{\mu\to\omega} \| Y(\mu) \|_{\mathfrak{H}} = 0. \qquad (4.43)$$

uniformly in $\omega$ in $\mathscr{I}$. In fact, this holds with reference to the Hilbert–Schmidt norm, which is seen from Lemma 4.4 and definitions (4.13), (4.24), and (4.41).

To illustrate the proof of relation (4.37), thirdly we show that

$$\sup_{\mu\in\mathscr{R}_{\pm}(\mathscr{I})} \| W(\mu) \|_{\mathfrak{H}} < \infty. \qquad (4.44)$$

If the operator $M(n)^{1/2}$ were a bounded similarity definitions (4.40) and (4.41) would imply that

$$[I - Y(\mu)]^{-1} = W(\mu).$$

It is an interesting fact, stated and proved in the technical report [38] that although $M(n)^{1/2}$ is not a bounded similarity this relation is still valid. Relations (4.43) and (4.45) together establish the validity of estimate (4.44). Finally inserting relations (4.43) and (4.44) in relation (4.42) we arrive at the validity of the key relation (4.37). This, in turn, establishes the validity of Condition $G_3(\mathscr{I})(\alpha)$.

To verify Condition $G_3(\mathscr{I})(\beta)$ we employ the key relation (4.37) again. It shows that it suffices to prove that

$$\lim_{\mu_1\to\omega}\lim_{\mu_1\to\omega} \| [S_0(\mu_1)]_{\mathfrak{S}} - [S_0(\mu_2)]_{\mathfrak{S}} \| = 0, \qquad (4.46)$$

uniformly in $\omega$ in $\mathscr{I}$. Replacing the operator $S_0(\mu)_{\mathfrak{H}}$ in estimate (4.26) by $S_0(\mu_1)_{\mathfrak{H}} - S_0(\mu_2)_{\mathfrak{H}}$ yields

$$\| [S_0(\mu_1) - S_0(\mu_2)]_{\mathfrak{S}} \| \leqslant \| M(n^{1/2})[S(\mu_1) - S_0(\mu_2)]M(n^{1/2})_{\mathfrak{H}} \|.$$

Combining this relation with Lemma 4.4 and relation (4.29) we arrive at the validity of relation (4.46).

To verify Condition $G_3(\mathscr{I})(\gamma)$ insert definition (4.35) in definition (3.8). This yields

$$T(\mu) = M(p - q(\text{Re } \mu))R_0(\mu).$$

Combining this with formula (4.30) we obtain

$$I - T(\mu) - Q^{-1}(\mu) = M(p - q(\text{Re } \mu))(S_0(\mu) - R_0(\mu)) + M(q(\text{Re } \mu) - q(\mu))S_0(\mu).$$

Inserting formula (4.38) and definitions (4.33), (4.40), and (4.41) in this relation we arrive at

$$M(1/n)^{1/2}\{I - T(\mu) - Q(\mu)^{-1}\}M(n^{1/2}) = (Y(\mu) - X(\mu))W(\mu)Y(\mu) - Y(\mu).$$

Hence we see from relations (4.34), (4.43), and (4.44) that

$$\lim_{\mu\to\omega} \| M(1/n)^{1/2}\{I - T(\mu) - Q(\mu)^{-1}\} M(n)^{1/2}_{\mathfrak{H}} \| = 0,$$

uniformly in $\omega$ in $\mathscr{I}$. This relation, together with the unitary equivalence relation (4.31) yields

$$\lim_{\mu\to\omega} \|\{I - T(\mu) - Q(\mu)^{-1}\}_{\mathfrak{G}} \| = 0.$$

This establishes the validity of Condition $G_3(\mathscr{I})(\gamma)$.

(d) Condition $A(\mathscr{I})$. Since unitary equivalence preserves compactness the proof of relation (4.34) shows that the operator

$$(I - Q(\mu)^{-1})_{\mathfrak{G}} = M(p - q(\mu))S_0(\mu)_{\mathfrak{G}}$$

is compact. This implies that each of the two limit operators, $(I - Q^{\pm}(\omega)^{-1})_{\mathfrak{G}}$, is also compact. In view of the compactness of this operator and the Fredholm alternative it suffices to show that the limit operator is one to one. This is the statement of the lemma that follows.

LEMMA 4.5. *Suppose that the potential p satisfies Condition S(T–K–W) and that the potential* $p_2$ *satisfies Condition* $O(\mathscr{I})$. *Suppose, further, that* $\omega \in \mathscr{I} \subset \mathscr{R}^+$ *is an exceptional point and h in* $\mathfrak{G}$ *is a corresponding exceptional vector. That is to say*

$$(I - M(p - q(\omega))S_0^{+}(\omega))_{\mathfrak{G}}h = 0 \quad or \quad (I - M(p - q(\omega))S_0^{-}(\omega))_{\mathfrak{G}}h = 0. \tag{4.48}$$

*Then*

$$h = 0. \tag{4.49}$$

To establish conclusion (4.49) recall definition (4.25) and relation (4.29). They show that $h$ in $\mathfrak{G}$ implies that $h$ is also in $\mathfrak{L}_1(\mathscr{R}^+)$. This, in turn, together with Lemma 4.4 and the Lebesgue theorem on dominated convergence implies that each of the two limits do exist,

$$g^{\pm}(\xi) = \lim_{\epsilon\to+0} S_0(\omega + i\epsilon)\, h(\xi).$$

At the same time it follows from formula (4.19) and definition (4.23) that this limit is given by

$$g^{\pm}(\xi) = \frac{y_\infty^{\pm}(\omega)(\xi)}{W(y_\infty^{\pm}(\omega), y_0^{\pm}(\omega))} \int_0^{\xi} y_0^{\pm}(\omega)(\eta)\, h(\eta)\, d\eta$$

$$+ y_0^{\pm}(\omega)(\xi) \int_\xi^{\infty} \frac{y_\infty^{\pm}(\omega)(\eta)}{W(y_\infty^{\pm}(\omega), y_0^{\pm}(\omega))} h(\eta)\, d\eta. \tag{4.50}$$

Here we have set

$$y_0^{\pm}(\omega) = \lim_{\epsilon \to 0} y_0(\omega \pm i\epsilon) \quad \text{and} \quad y_\infty^{\pm}(\omega) = \lim_{\epsilon \to 0} y_\infty(\omega \pm i\epsilon). \quad (4.51)_{0,\infty}^{\pm}$$

According to the basic Lemma 3.3

$$\lim_{\epsilon \to 0} [(h, S_0(\omega + i\epsilon)\, h) - (h, S_0(\omega - i\epsilon)\, h)] = 0. \tag{4.52}$$

Next we show that this implies the key relation.

$$\int_0^{\infty} y_0^{\pm}(\omega)(\eta)\, h(\eta)\, d\eta = 0. \tag{4.53}$$

To show this, in turn, we need a simple fact about the jump of a family of solutions of a differential equation. Specifically let $z_0(\mu)$ be a family of solutions of Eq. (4.18) which depend continuously on $\mu$ in the union of the two closed regions $\overline{\mathscr{R}_{\pm}(\mathscr{I})}$ and are such that

$$z_0(\bar{\mu}) = \overline{z_0(\mu)}.$$

Then let $z_\infty(\mu)$ be another family of solutions which depend continuously on $\mu$ in the union of these two open regions. Suppose that this second family is normalized with reference to the first one so that

$$W(z_0(\mu), z_\infty(\mu)) = 1.$$

Then at each real point $\omega$ the jump is given by

$$[z_\infty(\omega)]_-^+ = -W(z_\infty^+(\omega), z_\infty^-(\omega)) z_0(\omega). \tag{4.54}$$

According to definition (4.13) the approximate potential is such that,

$$q(\bar{\mu}) = \overline{q(\mu)},$$

and according to definition $(4.17)_0$ the solution $y_0(\mu)$ satisfies the real boundary condition $(2.5)_0$. These two facts allow us to set

$$z_0(\mu) = \frac{y_0(\mu)}{Dy_0(\mu)(0)}.$$

Clearly we can also set

$$z_\infty(\mu) = \frac{y_\infty(\mu)\, Dy_0(\mu)(0)}{W(y_0(\mu), y_\infty(\mu))}.$$

Then formula (4.54) yields

$$\left[\frac{y_\infty(\omega)\, Dy_0(\omega)(0)}{W(y_0(\omega), y_\infty(\omega))}\right]_-^+ = W_{\infty\infty}^{+-}(\omega)\left(\frac{y_0(\omega)}{-Dy_0(\omega)(0)}\right),$$

where

$$W_{\infty\infty}^{+-}(\omega) = W\left(\frac{y_\infty{}^+(\omega)\, Dy_0{}^+(\omega)(0)}{W(y_0{}^+(\omega), y_\infty{}^+(\omega))}; \frac{y_\infty{}^-(\omega)\, Dy_0{}^-(\omega)(0)}{W(y_0{}^-(\omega), y_\infty{}^-(\omega))}\right). \quad (4.55)$$

This formula, in turn, together with formula (4.19) and definition (4.23) yields

$$[S_0(\omega)(\xi, \eta)]_-^+ = W_{\infty\infty}^{+-}(\omega)\left(\frac{y_0(\omega)(\xi)}{Dy_0(\omega)(0)}\right)\left(\frac{y_0(\omega)(\eta)}{Dy_0(\omega)(0)}\right). \quad (4.56)$$

According to definitions (4.17), (4.16), and (4.51) for each positive $\omega$ the functions $y_\infty{}^+(\omega)$ and $y_\infty{}^-(\omega)$ have different asymptotic behavior near infinity. Therefore we see from definition (4.55) and from the assumption that $\omega$ is positive that this Wronskian is not zero. Combining this fact with relations (4.52) and (4.56) we arrive at the validity of the key relation (4.53).

The key relation (4.53), in turn, implies the validity of conclusion (4.49) in the usual manner. A possible reference for this implication is the technical report [38]. This completes the proof of Lemma 4.5.

Having established these conditions we can conclude from the abstract Theorem 3.1 the validity of the concrete Theorem 2.1. Again this follows in the usual manner and a possible reference is the technical report [38].

## References


1. E. P. Wigner and J. Von-Neumann, Über merkwürdige eigenwerte, *Z. Phys.* **30** (1929), 465, or "Collected Works of von-Neumann," (A. H. Taub, Ed.), pp. 550–552, Pergamon, New York, 1961. 550-552.
2. A. Wintner, Asymptotic integrations of the adiabatic oscillator, *Amer. J. Math.* **69** (1947), 251–272.
3. E. C. Titchmarsh, "Eigenfunction Expansions Associated with Second-Order Differential Equations," Clarendon, Oxford, 1948. See Section V.
4. K. Kodaira, The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of $S$-matrices, *Amer. J. Math.* **71** (1949), 921–945.
5. A. Ya, Povzner, On the expansion of an arbitrary function in terms of the eigenfunctions of the operator $-Au + cu$ (in Russian), *Mat. Sb.* **32** (1953), 109–156.
6. T. Ikebe, Eigenfunction expansion associated with the Schrödinger operators and their application to scattering theory, *Arch. Rational Mech. Anal.* **5** (1960), 1–34.
7. D. M. Eidus, The principle of limiting absorption, *Amer. Math. Soc. Transl.* **47** (1965), 157–192.

8. P. A. REJTO, On partly gentle perturbations, I. *J. Math. Anal. Appl.* **17** (1967), 453–462. (a) Property [*] and the proof of Lemma 2.1. (b) Theorem 4.2.
9. J. WEIDMANN, Zur Spektraltheorie von Sturm–Liouville-Operatoren, *Math. Z.* **98** (1967), 268–302. See Satz 5.1 and Korollar 5.2.
10. J. S. HOWLAND, A perturbation theoretic approach to eigenfunction expansions. *J. Math. Anal. Appl.* **20** (1967), 145–187.
11. B. SIMON, On positive eigenvalues of one body Schrödinger operators, *Comm. Pure Appl. Math.* **22** (1967), 531–538.
12. C. GOLDSTEIN, Eigenfunction expansions associated with the Laplacian for certain domains with infinite boundaries, I, II, and III, *Trans. Amer. Math. Soc.* **135** (1969), 1–31, 33–50; **143** (1969), 283–301.
13. W. JÄGER, Ein gewöhnlicher Differentialoperator zweiter Ordnung für Funktionen mit Werten in einem Hilbertraum, *Math. Z.* **113** (1970), 68–98.
14. T. KATO AND S. T. KURODA, On the abstract theory of scattering, *Rocky Mountain J. Math.* **1** (1971), 127–171.
15. W. O. AMREIN, V. GEORGESCU, AND J. M. JAUCH, Stationary state scattering theory, *Helv. Phys. Acta* **44** (1971), 407–434.
16. S. AGMON, Spectral theory of self-adjoint elliptic operators in $L_2(R^n)$, *in* "Abstracts of the 1971 Oberwolfach Symposium on the Mathematical Theory of Scattering."
17. S. T. KURODA, Scattering theory for differential operators, I, operator theory, *J. Math. Soc. Japan* **25** (1973), 75–104; Part II, preprint.
18. T. IKEBE AND Y. SAITO, Limiting absorption method and absolute continuity for the Schrödinger operator, *J. Math. Kyoto Univ.* **12** (1972), 513–542.
19. P. A. REJTO, On a theorem of Titchmarsh–Weidmann concerning absolutely continuous operators, I, *Res. Rep. Univ. Minn.* (1972).
20. E. PRUGOVEČKI, "Integral Representations of the Wave and Transition Operators in Non-Relativistic Scattering Theory," Univ. of Toronto (preprint).
21. R. LAVINE, Absolute continuity of the positive spectrum for Schrödinger operators with long range potentials, *J. Functional Analysis* **12** (1973), 30–54. a. proof of Theorem 4.
22. E. PRUGOVEČKI AND J. ZORBAS, Modified Lippmann–Schwinger equations for two-body scattering with long range interactions, *J. Math. Phys.* **14** (1973).
23. J. M. COMBES, Dilation analytic perturbations, *in* "Proceedings of the 1973 Nato Summer School on Scattering Theory," to appear.
24. R. LAVINE, Commutator methods in scattering theory, *in* "Proceedings of the 1973 Nato Summer School on Scattering Theory," to appear.
25. E. G. KEMBLE, "The Fundamental Principles of Quantum Mechanics with Elementary Applications," Dover, New York, 1937. (a) Equations (21.11) and (21.12). (b) Appendix D.
26. E. A. CODDINGTON AND N. LEVINSON, "Theory of Ordinary Differential Equations," McGraw-Hill, New York, 1955. See Section III.8.
27. K. O. FRIEDRICHS, "Spectral Theory of Operators in Hilbert Space," Courant Institute Lecture Notes, 1959. See Section 20.
28. M. A. NEUMARK, "Lineare Differentialoperatoren," Akademie-Verlag, Berlin, 1960.
29. T. KATO, "Perturbation Theory for Linear Operators," Springer–Verlag, Berlin, 1966. (a) Theorem V.4.3; (b) Subsection X.5.7.
30. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators," Parts I and II, Wiley–Interscience, New York, 1963 and 1967.
31. R. G. NEWTON, "Scattering Theory of Waves and Particles," McGraw–Hill, New York, 1966. See Section 18.2, the WKB method.
32. M. SCHECHTER, "Spectra of Partial Differential Operators," American Elsevier, New York, 1971. See Theorems 3.1, 2.2, 2.1, in Chapter 5 and their proofs.

33. K. E. GUSTAFSON AND P. A. REJTO, Some essentially self-adjoint Dirac operators with spherically symmetric potentials. *Israel J. Math.* **14** (1973), 63–75. See Theorem 3.1 and also Lemma 1.1 in *Pacific J. Math.* **19** (1966), 113.
34. V. GEORGESCU, Ph.D. Thesis, Univ. Geneva, to appear.
35. K. JÖRGENS AND J. WEIDMAN, "Spectral Properties of Hamiltonian Operators," Lecture Notes, Springer–Verlag, Berlin, 1973. See Corollary 3.4 and also Lemma 1.1 in *Pacific J. Math.* **19** (1966), 113.
36. M. SCHECHTER, "Principles of Functional Analysis," Academic Press, New York, 1971.
37. J. R. HOEGH-KROHN, Partly gentle perturbation with application to perturbation by annihilation–creation operators, *Proc. Nat. Acad. Sci.* **58** (1967), 2187–2192.
38. P. A. REJTO, "On a Theorem of Titchmarsh–Kodaira–Weidmann Concerning Absolutely Continuous Operators," Univ. Minn. Tech. Report, June 1974.